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**Math 564: Adv. Analysis 1** HOMEWORK 4 **Due: Nov 3 (Fri), 11:59pm**

Unless otherwise is specified, $(X,\mu)$ is a measure space and $L^1 := L^1(X,\mu)$.

**1.** (a) Let $(X,\mathcal{B})$ and $(Y,\mathcal{C})$ be measurable spaces and let $T : X \to Y$ be a $(\mathcal{B},\mathcal{C})$-measurable map. Prove the **change of variable formula**: for each measure $\mu$ on $\mathcal{B}$ and a measurable $f \in L^1(Y,\mathcal{C},T_*\mu)$,

$$\int_X (f \circ T)\, d\mu = \int_Y f\, d(T_*\mu).$$

(b) As an application, let $T_A : \mathbb{R}^d \to \mathbb{R}^d$ be the linear transformation given by a $d \times d$ invertible matrix $A$, i.e. $Tx := Ax$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}^d$ and prove that $T_*\lambda = |\det A|^{-1}\lambda$, to conclude that

$$\int (f \circ T_A)\, d\lambda = \int f |\det A|^{-1}\, d\lambda.$$

**2.** For $f_n \in L^1$, prove that if $\sum_{n\in\mathbb{N}} \int |f_n|\, d\mu < \infty$, then $\sum_{n\in\mathbb{N}} f_n(x)$ converges absolutely for a.e. $x \in X$ and $\sum_{n\in\mathbb{N}} f_n \in L^1$. Furthermore, the convergence is also in $L^1$, i.e. the partial sums $\sum_{n<N} f_n$ converge in the $L^1$-norm to $\sum_{n\in\mathbb{N}} f_n$. In particular:

$$\int \sum_{n\in\mathbb{N}} f_n\, d\mu = \sum_{n\in\mathbb{N}} \int f_n\, d\mu.$$

**3.** Prove the **generalized dominated convergence theorem**: Let $f_n, f \in L^1$ be such that $f_n \to f$ a.e. If there are non-negative $g_n, g \in L^1$ such that $g_n \to g$ a.e., $\int g_n\, d\mu \to \int g\, d\mu$, and $|f_n| \leqslant g_n$ for each $n \in \mathbb{N}$, then $f_n \to_{L^1} f$. In particular, $\int f_n\, d\mu \to \int f\, d\mu$.

**4.** Let $f_n, f \in L^1$ be such that $f_n \to f$ a.e. and $\int |f_n| \to \int |f|$.

(a) Prove that $f_n \to_{L^1} f$.

(b) Conclude that $\int_A f_n \to \int_A f$ for each measurable $A \subseteq X$.

**5.** Let $f_n \in L^1(\mathbb{R},\lambda)$ be a non-negative Lebesgue integrable functions on $\mathbb{R}$. Prove or give a counterexample to the following statements.

(a) $\displaystyle\int \limsup_{n\to\infty} f_n \geqslant \limsup_{n\to\infty} \int f_n$.

(b) If $f_n \to 0$ both pointwise and in the $L^1$-norm, then there is $g \in L^1(\mathbb{R},\lambda)$ such that $f_n \leqslant g$ for each $n \in \mathbb{N}$.

**6.** Let $(f_n)$ be a sequence of $\mu$-measurable functions $X \to \mathbb{R}$.

(a) Observe that if $(f_n)$ converges in measure (to some measurable function $f$) then it is Cauchy in measure.

(b) If $(f_n)$ is Cauchy in measure and a subsequence $(f_{n_k})$ converges in measure to a measurable function $f$, then $f_n \to_\mu f$.

**7. Tightness of the Fubini–Tonelli hypothesis.**

(a) **Non-$\sigma$-finite.** Let $X := [0,1]$, and let $\mu$ and $\nu$ be, respectively, Lebesgue and counting measures on $X$. Because $\nu$ is not $\sigma$-finite, the product measure $\mu \times \nu$ is not unique, but we let $\mu \times \nu$ denote the largest of them, namely, the outer measure $(\mu \times \nu)^*$ induced by the premeasure $\mu \times \nu$ on the algebra generated by $\mathcal{B}(X) \times \mathcal{B}(X)$.

For the diagonal $\Delta := \{(x,x) : x \in X\}$, compute

$$\int \nu(\Delta_x)\, d\mu(x), \quad \int \mu(\Delta^y)\, d\nu(y), \text{ and } \mu \times \nu(\Delta)$$

to verify that no two are equal to each other.

Hint: To compute $\mu \times \nu(\Delta)$, recall the definition of the outer measure.

(b) **Non-integrable.** Let $\mu = \nu$ be the counting measure on $\mathbb{N}$ (defined on $\mathscr{P}(\mathbb{N})$). Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by setting $f(x,y)$ to be 1 if $x = y$, $-1$ if $x = y+1$, and 0 otherwise. Verify that $f$ is not $\mu \times \nu$-integrable, and although the integrals $\int\int f\, d\mu\, d\nu$ and $\int\int f\, d\nu\, d\mu$ both exist, they are unequal.

(c) [*Optional*] **Non-measurable.** Let $X := \omega_1$ and let $\mathcal{M} = \mathcal{N}$ be the $\sigma$-algebra of countable and co-countable subsets of $\omega_1$. Let $\mu = \nu$ be defined to be 0 on countable and 1 on co-countable sets. Let $R :=<$, i.e. $\mathbb{R} := \{(x,y) \in \omega_1 \times \omega_1 : x < y\}$. Then $R$ is not in $\mathcal{M} \times \mathcal{N}$. However, the fibers $R_x$ and $R^y$ are in $\mathcal{M} = \mathcal{N}$ for each $x, y \in \omega_1$, and the integrals $\int \nu(R_x)\, d\mu(x)$ and $\int \mu(R^y)\, d\nu(y)$ exist, but they are unequal.

**8. $\mu\times\nu$-measurable Fubini–Tonelli.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces and let $f : X \times Y \to \overline{\mathbb{R}}$ be a $\mu \times \nu$-measurable function. Prove:

(a) The fibers $f_x$ and $f^y$ are respectively $\nu$ and $\mu$ measurable for $\mu$-a.e. $x \in X$ and $\nu$-a.e. $y \in Y$.

Caution: Folland claims $\mathcal{M}$ and $\mathcal{N}$ measurability instead, which is wrong.

(b) If $f \geqslant 0$, then functions $g : x \mapsto \int f_x\, d\nu$ and $h : y \mapsto \int f^y\, d\mu$ are defined respectively $\mu$-a.e. and $\nu$-a.e., and they are respectively $\mu$ and $\nu$ measurable. Moreover,

$$\int\int f\, d\nu\, d\mu = \int f\, d\mu \times \nu = \int\int f\, d\mu\, d\nu. \tag{$*$}$$

(c) If $f$ is $\mu \times \nu$-integrable, then in fact $g$ and $h$ are $\mu$ and $\nu$ integrable and ($*$) holds.